Revisiting the Simulink Library with Complementarity Conditions

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From Complementarity Conditions to Nonsmooth Dynamical Systems

Infusing Complementarity Conditions in the Standard Simulink Library
A Simulink Model
Stops at $t=10$
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Infusing Complementarity Conditions in the Standard Simulink Library
The Power of Complementarity Conditions

**Scalar form:** \(0 \leq \lambda \perp \mu \geq 0\) with \(\lambda, \mu\) scalar

\[\lambda, \mu \geq 0\ \text{and} \ \lambda \mu = 0\]

**General form:** \(\vec{\lambda} = \lambda_1 \ldots \lambda_n\) and \(\vec{\mu} = \mu_1 \ldots \mu_n\)

\(0 \leq \vec{\lambda} \perp \vec{\mu} \geq 0\) means

\[\forall i = 1 \ldots n, \ 0 \leq \lambda_i \perp \mu_i \geq 0\]

**Multibody mechanics:**

\[0 \leq h \perp f \geq 0\]

**Electronics:**

\[i = I_s (e^{-\frac{u}{U_s}} - 1)\]

\[0 \leq u - U_s \perp i \geq 0\]
The Power of Complementarity Problems

Example: \( y = \text{abs}(x) \)

\[
\begin{align*}
  y &= u + v \\
  x &= u - v \\
  0 &\leq u \perp v \geq 0
\end{align*}
\]

\( \partial f(x) = \{ y | \forall z, f(z) - f(x) \geq y(z-x) \} \)

Example: \( y = \text{sgn}(x) \)

\[
\begin{align*}
  x &= u - v \\
  0 &\leq u \perp v \geq 0 \\
  0 &\leq v \perp 1 - y \geq 0 \\
  0 &\leq v \perp 1 + y \geq 0
\end{align*}
\]

\( \text{sgn}(x) = \partial \text{abs}(x) \)
Linear Complementarity Problems & Systems

**Definition:** Linear Complementarity Problem (LCP)

\[
\begin{cases}
\lambda = M\mu + q \\
0 \leq \lambda \perp \mu \geq 0
\end{cases}
\]

**Theorem:** A LCP admits a unique solution for all \( q \) iff \( M \) is a P-matrix

**Definition:** \( M \) is a P-matrix iff all its principal minors \( \det([M]_{i,i}) \) are strictly positive

**Definition:** Linear Complementarity System = Linear ODE + LCP

\[
\dot{q} = Aq + B\lambda
\]
Zero-crossing as a complementarity system

\[
\begin{align*}
    y(t_0) &= 0 \\
    x &= u - v \\
    0 &\leq u \perp v \geq 0 \\
    0 &\leq y' \perp v \geq 0 \quad \text{// As long as } x < 0, \text{ } y \text{ is constant } = 0 \\
    0 &\leq 1 - y \perp u + y \geq 0 \quad \text{// As soon as } x > 0, \text{ } y = 1 \text{ and stays there}
\end{align*}
\]

**Remark:** nondeterministic if \( x \) reaches and stays at 0

**What is an event?** switching from saturation of \( u \geq 0 \) to saturation of \( v \geq 0 \) in \( 0 \leq u \perp v \geq 0 \)
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Simulink Library Blocks: the Easy Cases

Coulomb and Viscous Friction:

\[
\begin{align*}
y &= Gx + Fs \\
x &= u - v \\
0 &\leq 1 - s \perp u \geq 0 \\
0 &\leq 1 + s \perp v \geq 0
\end{align*}
\]

Dead Zone:

\[
\begin{align*}
y &= k - v \\
x &= u - v + L \\
0 &\leq u \perp v \geq 0 \\
0 &\leq k \perp l \geq 0
\end{align*}
\]
More Simulink Library Blocks

Saturation Integrator:

\[
\begin{aligned}
\dot{y} &= x + k - l \\
0 &\leq y - L \perp k \geq 0 \\
0 &\leq U - y \perp l \geq 0
\end{aligned}
\]

Backlash:

\[
\begin{aligned}
\dot{y} &= k - l \\
0 &\leq y - x + \frac{B}{2} \perp k \geq 0 \\
0 &\leq x - y + \frac{B}{2} \perp l \geq 0
\end{aligned}
\]